2020S1

QUESTION 3. **(10 marks)**

Show that

$$\frac{n}{2} < \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1}}_{= \sum_{i=1}^{2^n-1} \frac{1}{i}} < n$$

for all integers $n \geq 2$.

Step 1: show that $\sum_{i=1}^{2^n-1} \frac{1}{i} > \frac{n}{2}$ for all integers $n \geqslant 2$.

Base step: when $n=2$, $LHS = \sum_{i=1}^{3} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$RHS = 1$.

$LHS > RHS$. Hence, the inequality holds when $n=2$.

Inductive step: assume that the inequality holds when $n=k$ $(k \geqslant 2)$,

i.e. $\sum_{i=1}^{2^k-1} \frac{1}{i} > \frac{k}{2}$. When $n = k+1$,

$$\begin{aligned} LHS - RHS = \sum_{i=1}^{2^{k+1}-1} \frac{1}{i} - \frac{k+1}{2} &= \left(\sum_{i=1}^{2^k-1} \frac{1}{i}\right) + \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - \frac{k+1}{2} \\ &> \frac{k}{2} + \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - \frac{k+1}{2} \\ &= \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - \frac{1}{2}. \end{aligned}$$

by the induction hypothesis →

this is a sum of 2^k numbers

To get some intuition, let's take a look at what this term is when $k=3$:

$$\sum_{i=1}^{2^{3+1}-1} \frac{1}{i} = \underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{= \sum_{i=1}^{2^3-1} \frac{1}{i}} + \underbrace{\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15}}_{= \sum_{i=2^3}^{2^{3+1}-1} \frac{1}{i}}$$

Question: why is $\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} > \frac{1}{2}$?

Answer: there are 8 terms, each one is larger than $\frac{1}{16}$.

Since for all $i = 2^k, 2^k+1, 2^k+2, \ldots, 2^{k+1}-1$,

$\frac{1}{i} > \frac{1}{2^{k+1}}$,

$$LHS - RHS > \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - \frac{1}{2} > \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{2^{k+1}}\right) - \frac{1}{2} = \frac{2^k}{2^{k+1}} - \frac{1}{2} = 0.$$

Hence, $LHS > RHS$ and the inequality holds for $n = k+1$.

Therefore, by mathematical induction, $\sum_{i=1}^{2^n-1} \frac{1}{i} > \frac{n}{2}$ for all integers $n \geq 2$.

Step 2: show that $\sum_{i=1}^{2^n-1} \frac{1}{i} < n$ for all integers $n \geq 2$.

Base step: when $n=2$, $LHS = \sum_{i=1}^{3} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$RHS = 2$.

$LHS < RHS$. Hence, the inequality holds when $n=2$.

Inductive step: assume that the inequality holds when $n=k$ $(k \geq 2)$,

i.e. $\sum_{i=1}^{2^k-1} \frac{1}{i} < k$. When $n = k+1$,

$$LHS - RHS = \sum_{i=1}^{2^{k+1}-1} \frac{1}{i} - (k+1) = \left(\sum_{i=1}^{2^k-1} \frac{1}{i}\right) + \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - (k+1)$$

by the induction hypothesis $\rightarrow$ $< k + \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - (k+1)$

$$= \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - 1.$$

Again, let's take a look at what this term is when $k=3$.

$$\sum_{i=1}^{2^{3+1}-1} \frac{1}{i} = \underbrace{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}_{=\sum_{i=1}^{2^3-1}\frac{1}{i}} + \underbrace{\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}}_{=\sum_{i=2^3}^{2^{3+1}-1}\frac{1}{i}}$$

Question: why is $\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15} \leq 1$?

Answer: there are 8 terms, each one is less than or equal to $\frac{1}{8}$.

Since for all $i = 2^k, 2^k+1, 2^k+2, \ldots, 2^{k+1}-1$,

$$\frac{1}{i} \leq \frac{1}{2^k}$$

$$LHS - RHS < \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i}\right) - 1 \leq \left(\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{2^k}\right) - 1 = \frac{2^k}{2^k} - 1 = 0.$$

Hence, $LHS < RHS$ and the inequality holds for $n = k+1$.

Therefore, by mathematical induction, $\sum_{i=1}^{2^n-1} \frac{1}{i} < n$ for all integers $n \geq 2$.