Week 9

Q15: This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Proof that LHS $\subseteq$ RHS : let $x \in$ LHS be arbitrary. Then, $x \in A \cup B$ and $x \notin A \cap B$.

Case 1 : $x \in A$. Then, $x \notin B$ because $x \notin A \cap B$. Since $x \in A - B \subseteq$ RHS, $x \in$ RHS.

Case 2 : $x \in B$. Then, $x \notin A$ because $x \notin A \cap B$. Since $x \in B - A \subseteq$ RHS, $x \in$ RHS.

In both cases, $x \in$ RHS holds. Therefore, LHS $\subseteq$ RHS.

Proof that RHS $\subseteq$ LHS : let $x \in$ RHS be arbitrary. Then, either $x \in A - B$ or $x \in B - A$.

Case 1 : $x \in A - B$. Then, $x \in A$ and $x \notin B$. Thus, $x \in A \cup B$ and $x \notin A \cap B$. This shows that $x \in (A \cup B) \cap \overline{A \cap B}$ = LHS.

Case 2 : $x \in B - A$. Then, $x \in B$ and $x \notin A$. Thus, $x \in A \cup B$ and $x \notin A \cap B$. This shows that $x \in (A \cup B) \cap \overline{A \cap B}$ = LHS.

In both cases, $x \in$ LHS holds. Therefore, RHS $\subseteq$ LHS.

We conclude that LHS = RHS.

Q16: The symmetric difference of A and B, denoted by $A \triangle B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \triangle B) \triangle B = A$ by showing that each side of the equation is a subset of the other.

Notice that $A \triangle B = (A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$.

Proof that LHS $\subseteq$ RHS: let $x \in$ LHS be arbitrary. Then,

$x \in (A \triangle B) \triangle B = ((A \triangle B) - B) \cup (B - (A \triangle B))$.

Case 1: $x \in (A \triangle B) - B$. Then, $x \in A \triangle B \subseteq A \cup B$ and $x \notin B$.

Hence, we get $x \in A$.

Case 2: $x \in B - (A \triangle B)$. We will try to simplify $B - (A \triangle B)$:

$$
\begin{aligned}
B - (A \triangle B) &= B \cap \overline{A \triangle B} && \text{(alt. rep. of "$-$")} \\
&= B \cap \overline{(A-B) \cup (B-A)} && \text{(by the definition of "$\triangle$")} \\
&= B \cap \overline{A-B} \cap \overline{B-A} && \text{(De Morgan)} \\
&= B \cap \overline{A \cap \overline{B}} \cap \overline{B \cap \overline{A}} && \text{(alt. rep. of "$-$")} \\
&= [B \cap (\overline{A} \cup B)] \cap (\overline{B} \cup A) && \text{(De Morgan)} \\
&= B \cap (\overline{B} \cup A) && \text{(absorption)} \\
&= (B \cap \overline{B}) \cup (B \cap A) && \text{(distributivity)} \\
&= \phi \cup (B \cap A) \\
&= B \cap A.
\end{aligned}
$$

Since $B - (A \triangle B) = B \cap A$, we get $x \in B \cap A$. Hence, $x \in A$.

In both cases, $x \in A$ holds. Therefore, LHS $\subseteq$ RHS.

Proof that RHS $\subseteq$ LHS: let $x \in$ RHS $= A$ be arbitrary. Then, either $x \in B$ or $x \notin B$.

Case 1: $x \in B$. Then, $x \in A \cap B = B - (A \triangle B) \subseteq$ LHS. Thus, $x \in$ LHS.

Case 2: $x \notin B$. Then, $x \in A - B \subseteq A \triangle B$. Since $x \in A \triangle B$ and $x \notin B$, we get $x \in (A \triangle B) - B \subseteq$ LHS and thus $x \in$ LHS.

In both cases, $x \in$ LHS holds. Therefore, RHS $\subseteq$ LHS.

We conclude that LHS = RHS.

Q16: The symmetric difference of A and B, denoted by $A \triangle B$, is the set containing those elements in either A or B, but not in both A and B.

2. Prove that $(A \triangle B) \triangle B = A$ using a membership table.

A	B	$A-B$	$B-A$	$A \triangle B$	$(A \triangle B)-B$	$B-(A \triangle B)$	LHS
0	0	0	0	0	0	0	0
0	1	0	1	1	0	0	0
1	0	1	0	1	1	0	1
1	1	0	0	0	0	1	1

Since the columns corresponding the two sides of the identity are identical, the identity holds.

Alternatively, from columns 1, 2, 5 of the membership table above,

$A \triangle B = \{x : (x \in A) \text{ XOR } (x \in B)\}$, where XOR refers to "exclusive OR". Using the associativity of the "XOR" operator, we can get the associativity of the "$\triangle$" operator over sets. Specifically,

$$(A \triangle B) \triangle B = \{x : ((x \in A) \text{ XOR } (x \in B)) \text{ XOR } (x \in B)\}$$
$$= \{x : (x \in A) \text{ XOR } ((x \in B) \text{ XOR } (x \in B))\} = A \triangle (B \triangle B).$$

Consequently, $(A \triangle B) \triangle B = A \triangle (B \triangle B)$
$$= A \triangle \emptyset$$
$$= A.$$

Relations

$A = \{1,2,3\}$, $B = \{4,5,6,7\}$.

Relation as a set: $R \subseteq A \times B$. For all $x \in A$, $y \in B$, $xRy \Leftrightarrow (x,y) \in R$.

e.g. $R = \{(1,4), (1,5), (2,6), (3,7)\}$. Then, $1R4 \equiv T$, $2R6 \equiv T$, $3R6 \equiv F$, $1R7 \equiv F$.

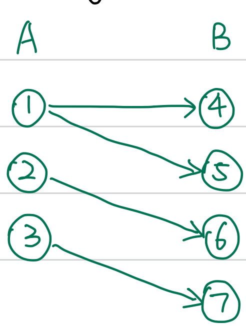
Relation as a matrix: represent elements of A as rows, represent elements of B as columns. Each entry is TRUE iff the corresponding elements are related.

e.g.

$$\begin{array}{c c} & \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} T & T & F & F \\ F & F & T & F \\ F & F & F & T \end{bmatrix} \end{array}$$

Relation as a directed graph: represent elements of A and B as vertices. $xRy \leftrightarrow$ there is an edge from x to y.

e.g.



Inverse : $\forall x \in A, \forall y \in B,\ xRy \leftrightarrow yR^{-1}x$

From a set perspective : $R^{-1} \subseteq B \times A,\ \forall x \in A, \forall y \in B,\ (x,y) \in R \leftrightarrow (y,x) \in R^{-1}$.

e.g. $R = \{(1,4),\ (1,5),\ (2,6),\ (3,7)\}$,

$R^{-1} = \{(4,1),\ (5,1),\ (6,2),\ (7,3)\}$.

From a matrix perspective : the matrix corresponding to R^{-1} is the <u>transpose</u> of the matrix corresponding to R.

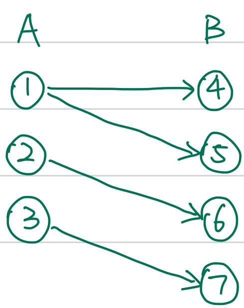
e.g.

$$R : \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} T & T & F & F \\ F & F & T & F \\ F & F & F & T \end{bmatrix} \end{array}, \quad R^{-1} : \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} T & F & F \\ T & F & F \\ F & T & F \\ F & F & T \end{bmatrix} \end{array}.$$

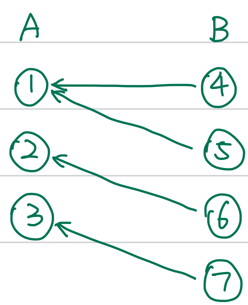
From a graph perspective : reversing the direction of all edges in the graph corresponding to R results in the graph corresponding to R^{-1}.

e.g.

R :



R^{-1} :



Q2: Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \Leftrightarrow x|y$. Compute the matrix of the inverse relation R^{-1}.

$x R y \Leftrightarrow x$ divides y

$y R^{-1} x \Leftrightarrow x R y \Leftrightarrow y$ is divisible by x.

Hence, R describes the "divides" relation, and R^{-1} describes the "is divisible by" relation.

The matrix of R is :

$$\begin{array}{c|ccc} & 2 & 6 & 8 \\ \hline 2 & T & T & T \\ 3 & F & T & F \\ 4 & F & F & T \end{array}$$

The matrix of R^{-1} is the transpose of the matrix of R :

$$\begin{array}{c|ccc} & 2 & 3 & 4 \\ \hline 2 & T & F & F \\ 6 & T & T & F \\ 8 & T & F & T \end{array}$$

that is, the first row of R becomes the first column of R^{-1},
the second row of R becomes the second column of R^{-1},
the third row of R becomes the third column of R^{-1}.

Composition: R is a relation from A to B, S is a relation from B to C. $S \circ R$ is a relation from A to C.

$$\forall x \in A, \forall z \in C, \quad x(S \circ R)z \leftrightarrow \exists y \in B, \; xRy \wedge ySz.$$

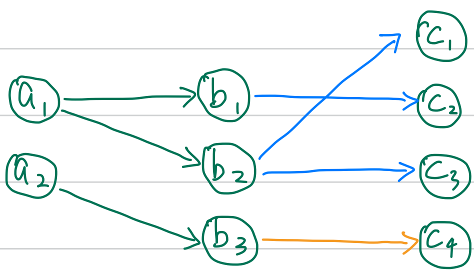
From a set perspective:

$$S \circ R = \{(x,z) \in A \times C : \exists y \in B, (x,y) \in R, (y,z) \in S\}.$$

From a graph perspective:

For every pair $(x,z) \in A \times C$, $x(S \circ R)z$ if and only if there is a path from x to z (through an element in B)

e.g. A B C



c_1 is reachable from a_1 $\Rightarrow$ $a_1(S \circ R)c_1 \equiv T$

c_2 is reachable from a_1 $\Rightarrow$ $a_1(S \circ R)c_2 \equiv T$

c_3 is NOT reachable from a_2 $\Rightarrow$ $a_2(S \circ R)c_3 \equiv F$

Composition from a matrix perspective:

Q4: This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

$$R = \{(a_1, b_1), (a_1, b_2)\}, \qquad S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

What is the matrix of $S \circ R$?

2. In general, what is the matrix of $S \circ R$?

The matrix of R:
$$M_R = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} T & T \\ F & F \end{bmatrix} \end{array}.$$

The matrix of S:
$$M_S = \begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} b_1 \\ b_2 \end{array} & \begin{bmatrix} T & F & T \\ T & T & F \end{bmatrix} \end{array}.$$

The matrix of $S \circ R$ is the matrix product of M_R and M_S,

where: $p + q$ (addition) is replaced by $p \vee q$ (disjunction),
$p \cdot q$ (multiplication) is replaced by $p \wedge q$ (conjunction).

$$M_R \cdot M_S = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} T & T \\ F & F \end{bmatrix} \end{array} \cdot \begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} b_1 \\ b_2 \end{array} & \begin{bmatrix} T & F & T \\ T & T & F \end{bmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} (T \wedge T) \vee (T \wedge T) & (T \wedge F) \vee (T \wedge T) & (T \wedge T) \vee (T \wedge F) \\ (F \wedge T) \vee (F \wedge T) & (F \wedge F) \vee (F \wedge T) & (F \wedge T) \vee (F \wedge F) \end{bmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} T & T & T \\ F & F & F \end{bmatrix} \end{array}.$$

Note that for two real matrices M_1 and M_2:

$$M_1 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \quad M_2 = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \end{bmatrix},$$

$$M_1 \cdot M_2 = \begin{bmatrix} p_{11}q_{11}+p_{12}q_{21} & p_{11}q_{12}+p_{12}q_{22} & p_{11}q_{13}+p_{12}q_{23} \\ p_{21}q_{11}+p_{22}q_{21} & p_{21}q_{12}+p_{22}q_{22} & p_{21}q_{13}+p_{22}q_{23} \end{bmatrix}.$$

In general, the entry of $M_1 \cdot M_2$ on the i-th row and j-th column is equal to $\left(\sum_{k=1,2} p_{ik}q_{kj}\right)$.

Try to compare the product of real matrices and the product of boolean matrices and convince yourself that the matrix of a composite relation is the product of the matrices of the individual relations.

(Will be discussed next week)

From now on, we consider relations from a set A to itself.

<u>Reflexivity</u>: R is reflexive $\Leftrightarrow$ $\forall x \in A,\ xRx$. (unconditional)

<u>Symmetry</u>: R is symmetric $\Leftrightarrow$ $\forall x \in A, \forall y \in A,\ xRy \to yRx$.

<u>Antisymmetry</u>: R is antisymmetric $\Leftrightarrow$ $\forall x \in A, \forall y \in A,\ xRy \wedge yRx \to x=y$.

<u>Transitivity</u>: R is transitive $\Leftrightarrow$ $\forall x \in A, \forall y \in A, \forall z \in A,\ xRy \wedge yRz \to xRz$.

From a set perspective :

Reflexivity : $\forall x \in A,\ (x,x) \in R$.

Symmetry : $\forall x \in A, \forall y \in A,\ (x,y) \in R \leftrightarrow (y,x) \in R$

$$\equiv \forall x \in A, \forall y \in A,\ \underbrace{\big((x,y) \in R \wedge (y,x) \in R\big)}_{\text{both are included}} \vee \underbrace{\big((x,y) \notin R \wedge (y,x) \notin R\big)}_{\text{neither is included}}.$$

Antisymmetry : $\forall x \in A, \forall y \in A,\ (x \neq y) \to \neg\big((x,y) \in R \wedge (y,x) \in R\big)$

$$\equiv \forall x \in A, \forall y \in A,\ (x \neq y) \to \underbrace{(x,y) \notin R \vee (y,x) \notin R}_{\text{at most one can be included}}$$

Transitivity : $\forall x \in A, \forall y \in A, \forall z \in A,\ (x,y) \in R \wedge (y,z) \in R \to (x,z) \in R$.

(not very insightful ...)

From a matrix perspective:

Reflexivity: diagonal entries are TRUE. Off-diagonal entries can be TRUE or FALSE (don't matter).

e.g.

$$\begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & T & T & F & T \\ a_2 & F & T & F & F \\ a_3 & F & T & T & T \\ a_4 & T & T & T & T \end{array}$$

Symmetry: entries above the diagonal and their "mirror reflections" below the diagonal need to be the same (both TRUE or both FALSE), diagonal entries can be TRUE or FALSE (don't matter).

e.g.

$$\begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & T & F & T & F \\ a_2 & F & F & F & F \\ a_3 & T & F & F & T \\ a_4 & F & F & T & T \end{array}$$

T — T & F — F

are allowed

T — F & F — T

are NOT allowed

In other words, the matrix is symmetric (i.e. equal to its transpose).

Antisymmetry: entries above the diagonal and their "mirror reflections" below the diagonal cannot both be TRUE (can be TRUE-FALSE, FALSE-TRUE, or FALSE-FALSE), diagonal entries can be TRUE or FALSE (don't matter).

e.g.

$$\begin{array}{c|cccc} & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & T & F & T & F \\ a_2 & T & F & F & F \\ a_3 & F & T & F & T \\ a_4 & F & T & F & T \end{array}$$

T — T & F — T & F — F are allowed

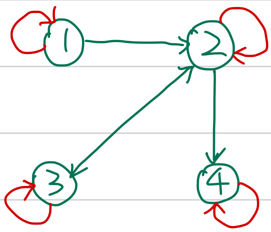
T — T is NOT allowed

Transitivity: not very insightful using the matrix perspective.

From a graph perspective:

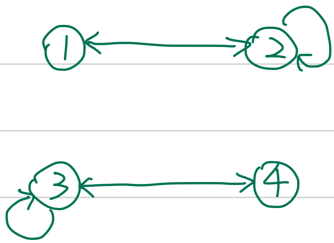
Reflexivity: every element has a self-loop.

e.g.



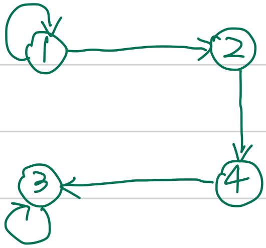
Symmetry: every edge is bidirectional unless it is a self-loop.

e.g.



Antisymmetry: there is NO bidirectional edge.

e.g.



Transitivity: not very insightful using the graph perspective.