Week 3

Recap

A <u>proposition</u> is a declarative statement that is either TRUE(T) or FALSE(F)

<u>Logical operators</u> combine propositions into compound proposition

$p \wedge q$	Conjunction	and
$p \vee q$	Disjunction	or
$\neg p$	Negation	not
$p \rightarrow q$	Conditional (Implication)	if p then q / p only if q
$p \leftrightarrow q$	Biconditional (Equivalence)	p if and only if (iff) q

$P \equiv Q \iff P$ and Q are <u>equivalent expressions</u>

$\iff P$ and Q have identical <u>truth tables</u>

<u>Tautology</u> : compound proposition that is always TRUE.

<u>Contradiction</u> : compound proposition that is always FALSE.

Conditional operator

p = I pay the rent $\qquad$ q = landlord fixes the ceiling

$p \to q$ $\equiv$ "If I pay the rent, then the landlord has fixed the ceiling"

$\equiv$ "I pay the rent only if the landlord has fixed the ceiling"

$\equiv$ "either the landlord has fixed the ceiling, or I do not pay the rent"

by conversion theorem → $\equiv \neg p \vee q$

rent paid	ceiling fixed	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Vacuously true!

p is sufficient for q

q is necessary for p

$q \to p \equiv$ "If the landlord has fixed the ceiling, then I pay the rent"

$\not\equiv p \to q$!

rent paid	ceiling fixed	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

$p \leftrightarrow q \equiv$ "I pay the rent if and only if the landlord has fixed the ceiling"

$\equiv (p \land q) \lor (\neg p \land \neg q)$

$\equiv (p \to q) \land (q \to p)$

rent paid	ceiling fixed	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p is sufficient and necessary for q

Original statement $p \to q$ $\equiv$ Contrapositive $\neg q \to \neg p$

Converse $q \to p$ $\equiv$ inverse $\neg p \to \neg q$

Suggestion: if you are still puzzled, try to come up with more examples.

Logical equivalence laws

$\neg T \equiv F$ $\quad$ $\neg F \equiv T$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\quad$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan)

$p \wedge q \equiv q \wedge p$ $\quad$ $p \vee q \equiv q \vee p$ (commutativity)

$\neg(\neg p) \equiv p$ (double negation)

$p \vee (p \wedge q) \equiv p$ $\quad$ $p \wedge (p \vee q) \equiv p$ (absorption)

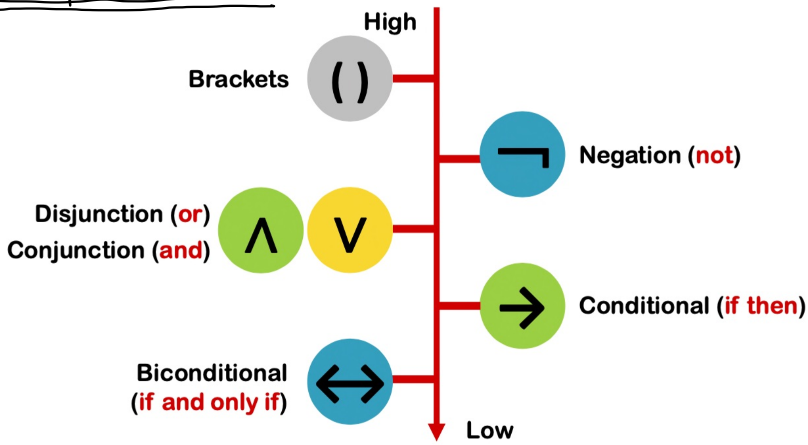
$p \wedge p \equiv p$ $\quad$ $p \vee p \equiv p$ (idempotent)

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (distributivity)

$p \to q \equiv \neg p \vee q$ (Conversion theorem)

It is <u>optional</u> to write down the name of the law.

Operator precedence



Suggestion: use brackets as much as possible for improved readability and for avoiding careless mistakes.

Argument

Premise 1 ;

Premise 2 ;

⋮

Premise n ;

$\therefore$ Conclusion

This argument is valid

$\Leftrightarrow$ $[(\text{Premise } 1 \wedge \cdots \wedge \text{Premise } n) \to \text{Conclusion}] \equiv T$

Showing the validity of an argument using truth table :

Critical row : rows in which all premises are TRUE,

i.e., $(\text{Premise } 1 \wedge \cdots \wedge \text{Premise } n) \equiv T$

Argument is valid $\Leftrightarrow$ Conclusion $\equiv T$ in all critical rows

notice the asymmetry

Argument is invalid $\Leftrightarrow$ There is a critical row in which Conclusion $\equiv F$ (a.k.a. Counterexample)

Non-critical rows are inconsequential and can be ignored (with appropriate explanations)

Remark 1 : Valid argument does NOT require that the premises are true!

Quote from Wikipedia:

"An argument is valid if and only if it would be contradictory for the conclusion to be false if all of the premises are true. Validity doesn't require the truth of the premises, instead it merely necessitates that conclusion follows from the formers without violating the correctness of the logical form. If also the premises of a valid argument are proven true, this is said to be sound."

An argument that is valid but not sound (from Wikipedia):

"All animals live on Mars. ← false premise
All humans are animals.
Therefore, all humans live on Mars."

Source : https://en.wikipedia.org/wiki/Validity_(logic)

Remark 2

The correct statement regarding the premises when determining the validity of an argument: since the validity of an argument depends entirely on the truth value of the conclusion **in the cases** (i.e. combinations of truth values of elementary propositions like p, q, r, s, t) **where all premises are true** (which correspond to the critical rows in the truth table), we only need to focus on these cases (which means that we only need to check the truth value of the conclusion in the critical rows and can ignore the non-critical rows).

Therefore, please do <u>NOT</u> write:
" Since $\neg p$ is a premise, $p \equiv F$ " in the exams!

Inference rules

Modus ponens

$$p \to q;$$
$$p;$$
$$\therefore q$$

Modus tollens

$$p \to q;$$
$$\neg q;$$
$$\therefore \neg p$$

Conjunctive simplification

$$p \wedge q;$$
$$\therefore p$$

Conjunctive addition

$$p;$$
$$q;$$
$$\therefore p \wedge q$$

Disjunctive addition

$$p;$$
$$\therefore p \vee q$$

Disjunctive syllogism

$$p \vee q;$$
$$\neg p;$$
$$\therefore q$$

Alternative rule of Contradiction

$$\neg p \to F;$$
$$\therefore p$$

Dilemma

$$p \vee q;$$
$$p \to r;$$
$$q \to r;$$
$$\therefore r$$

Hypothetical Syllogism

$$p \to q;$$
$$q \to r;$$
$$\therefore p \to r$$

To prove the validity of an argument via inference rule, one is adviced to use a succinct table format (see later).

Tips for exams

Questions related to logical equivalence (e.g. Q2 to Q8) and validity of argument (e.g. Q9 to Q12) are standard and easy to score.

They are standard because they all have fixed formats, and the two standard methods: (i) truth table & (ii) logical equivalence laws / inference rules always apply.

They are easy to score because there are two different ways to solve them. E.g. if you have used logical equivalence laws to solve a question, you can use truth table to solve it again to check your answer and make sure that you did not make a careless mistake.

In the exams, names of inference rules will be provided in footnote[1] without the exact formulas.

[1] The inference rules you may use are: modus ponens, modus tollens, conjunctive simplification, conjunctive addition, disjunctive addition, disjunctive syllogism, rule of contradiction and disjunction elimination.

Q4: These two laws are called distributivity laws. Show that they hold:

1. Show that $\underbrace{(p \wedge q) \vee r}_{LHS} \equiv \underbrace{(p \vee r) \wedge (q \vee r)}_{RHS}$.

p	q	r	$(p \wedge q)$	$(p \vee r)$	$(q \vee r)$	LHS	RHS
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

Same

Since the truth values of $(p \wedge q) \vee r$ and $(p \vee r) \wedge (q \vee r)$ are the same in all cases, $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$ holds.

Q5: Verify $\underbrace{\neg(p \vee \neg q) \vee (\neg p \wedge \neg q)}_{LHS} \equiv \underbrace{\neg p}_{RHS}$ by

- constructing a truth table,

p	q	$(p \vee \neg q)$	$\neg(p \vee \neg q)$	$(\neg p \wedge \neg q)$	LHS	RHS
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	F	T	F	T	T
F	F	T	F	T	T	T

same

Since the truth values of $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q)$ and $\neg p$ are the same in all cases, $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ holds.

- developing a series of logical equivalences.

$$\begin{aligned}
\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) &\equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q) && \text{(de Morgan)} \\
&\equiv \neg p \wedge (q \vee \neg q) && \text{(distributivity)} \\
&\equiv \neg p \wedge T && \text{(since } q \vee \neg q \equiv T\text{)} \\
&\equiv \neg p
\end{aligned}$$

$\neg p \wedge (q \vee \neg q) \equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

by the distributivity law

Q7: Show that $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$.

By showing LHS $\equiv \cdots \equiv$ RHS :

LHS $\equiv (p \vee q) \to r$

$\equiv \neg(p \vee q) \vee r$ (conversion theorem)

$\equiv (\neg p \wedge \neg q) \vee r$ (de Morgan)

$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ (distributivity)

$\equiv (p \to r) \wedge (q \to r)$ (conversion theorem)

$\equiv$ RHS

By showing RHS $\equiv \cdots \equiv$ LHS :

RHS $\equiv (p \to r) \wedge (q \to r)$

$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ (conversion theorem)

$\equiv (\neg p \wedge \neg q) \vee r$ (distributivity)

$\equiv \neg(p \vee q) \vee r$ (de Morgan)

$\equiv (p \vee q) \to r$ (conversion theorem)

$\equiv$ LHS

By showing $LHS \equiv \cdots \equiv IV$ and then showing $RHS \equiv \cdots \equiv IV$

(IV stands for intermediate value)

$$
\begin{aligned}
LHS &\equiv (p \vee q) \to r \\
&\equiv \neg(p \vee q) \vee r && \text{(conversion theorem)} \\
&\equiv (\neg p \wedge \neg q) \vee r && \text{(de Morgan)} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{(distributivity)}
\end{aligned}
$$

$$
\begin{aligned}
RHS &\equiv (p \to r) \wedge (q \to r) \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{(conversion theorem)}
\end{aligned}
$$

Q11: Determine whether the following argument is valid:

$$\begin{aligned}
&\neg p \to r \wedge \neg s; && \text{(P1)} \\
&t \to s; && \text{(P2)} \\
&u \to \neg p; && \text{(P3)} \\
&\neg w; && \text{(P4)} \\
&u \vee w; && \text{(P5)} \\
\therefore\ &t \to w.
\end{aligned}$$

Step	Formula	Reason/Rule
(1)	$u \vee w$	(P5)
(2)	$\neg w$	(P4)
(3)	u	(1)+(2), by disjunctive syllogism
(4)	$u \to \neg p$	(P3)
(5)	$\neg p$	(3)+(4), by modus ponens
(6)	$\neg p \to (r \wedge \neg s)$	(P1)
(7)	$r \wedge \neg s$	(5)+(6), by modus ponens
(8)	$\neg s$	(7), by conjunctive simplification
(9)	$t \to s$	(P2)
(10)	$\neg t$	(8)+(9), by modus tollens
(11)	$\neg t \vee w$	(10), by disjunctive addition
(12)	$t \to w$	by applying the conversion theorem to (11)

Therefore, the argument is valid.

Q12: Determine whether the following argument is valid:

$$\begin{array}{ll} p; & (P1) \\ p \vee q; & (P2) \\ q \to (r \to s); & (P3) \\ t \to r; & (P4) \\ \therefore \neg s \to \neg t. & \end{array}$$

(A quick inspection of the premises reveals that no known inference rule can be applied. Therefore, we will attempt to find a counterexample to show that the argument is invalid.)

Construction process of a counterexample:

(Recall that a counterexample is a combination of truth values of p, q, r, s, t under which all premises are TRUE and the conclusion is FALSE.)

To make the conclusion $(\neg s \to \neg t) \equiv F$, we require $s \equiv F$, $t \equiv T$. To make the first premise TRUE, we require $p \equiv T$. To make the fourth premise $(t \to r) \equiv T$ given that $t \equiv T$, we require $r \equiv T$ Given $r \equiv T$, $s \equiv F$, we know $(r \to s) \equiv F$. Hence, to make the third premise $(q \to (r \to s)) \equiv T$, we require $q \equiv F$. Given $p \equiv T$, $q \equiv F$, the second premise $(p \vee q) \equiv T$.

informal reasoning

We have identified a counterexample: $p \equiv T$, $q \equiv F$, $r \equiv T$, $s \equiv F$, $t \equiv T$ under which all premises are TRUE and the conclusion is FALSE

The corresponding row in the truth table looks like this:

p	q	r	s	t	(P1)	(P2)	(P3)	(P4)	Conclusion
T	F	T	F	T	T	T	T	T	F

All premises are TRUE, yet conclusion is FALSE. This is a counterexample.

Q12: Determine whether the following argument is valid:

$$p; \quad (P1)$$
$$p \vee q; \quad (P2)$$
$$q \to (r \to s); \quad (P3)$$
$$t \to r; \quad (P4)$$
$$\therefore \neg s \to \neg t.$$

Via truth table

p	q	r	s	t	(r→s)	(P1)	(P2)	(P3)	(P4)	Conclusion
T	T	T	T	T	T	T	T	T	T	T
T	T	T	T	F	T	T	T	T	T	T
T	T	T	F	T	F	T	T	F	T	F
T	T	T	F	F	F	T	T	F	T	T
T	T	F	T	T	T	T	T	T	F	T
T	T	F	T	F	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	F	F
T	T	F	F	F	T	T	T	T	T	T
T	F	T	T	T	T	T	T	T	T	T
T	F	T	T	F	T	T	T	T	T	T
T	F	T	F	T	F	T	T	T	T	F
T	F	T	F	F	F	T	T	T	T	T
T	F	F	T	T	T	T	T	T	F	T
T	F	F	T	F	T	T	T	T	T	T
T	F	F	F	T	T	T	T	T	F	F
T	F	F	F	F	T	T	T	T	T	T

Counter-example

Highlighted rows are critical. By the counterexample, the argument is invalid. Rows where $p \equiv F$ are omitted since in these rows $(P1) \equiv F$, which makes these rows non-critical.

Additional Questions

Determine whether the following argument is valid[1].

$$
\begin{array}{ll}
(p \wedge q) \to (r \vee s); & \text{(P1)} \\
\neg r; & \text{(P2)} \\
p \to q; & \text{(P3)} \\
p; & \text{(P4)} \\
\therefore s. &
\end{array}
$$

Via inference rules:

Step	Formula	Reason
(1)	$p \to q$	(P3)
(2)	p	(P4)
(3)	q	(1)+(2), by modus ponens
(4)	$p \wedge q$	(2)+(3), by conjunctive addition
(5)	$(p \wedge q) \to (r \vee s)$	(P1)
(6)	$r \vee s$	(4)+(5), by modus ponens
(7)	$\neg r$	(P2)
(8)	s	(6)+(7), by disjunctive syllogism

Therefore, the argument is valid.

Determine whether the following argument is valid[1].

$$
\begin{array}{ll}
(p \wedge q) \rightarrow (r \vee s); & \text{(P1)} \\
\neg r; & \text{(P2)} \\
p \rightarrow q; & \text{(P3)} \\
p; & \text{(P4)} \\
\therefore s. &
\end{array}
$$

Via truth table

p	q	r	s	(p ∧ q)	(r ∨ s)	(P1)	(P2)	(P3)	(P4)	Conclusion
T	T	F	T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	T	T	T	F
T	F	F	T	F	T	T	T	F	T	T
T	F	F	F	F	F	T	T	F	T	F

The highlighted row is the only critical row. Since the conclusion is TRUE in the only critical row, the argument is valid. Rows in which $p \equiv F$ or $r \equiv T$ have been omitted since $(P4) \equiv F$ whenever $p \equiv F$, and $(P2) \equiv F$ whenever $r \equiv T$. This means that these rows are non-critical.

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Translate into an argument (using logical symbols):

Let $s \equiv$ "it is sunny",

$c \equiv$ "it is colder than yesterday",

$w \equiv$ "we go swimming",

$t \equiv$ "we take a canoe trip",

$h \equiv$ "we will be home by sunset".

Argument:

$\neg s \wedge c$; (P1)

$w \to s$; (P2)

$\neg w \to t$; (P3)

$t \to h$; (P4)

$\therefore h$

Showing the validity via inference rules:

Step	Formula	Reason
(1)	$\neg s \wedge c$	(P1)
(2)	$\neg s$	(1), by Conjunctive simplification
(3)	$w \rightarrow s$	(P2)
(4)	$\neg w$	(2)+(3), by modus tollens
(5)	$\neg w \rightarrow t$	(P3)
(6)	t	(4)+(5), by modus ponens
(7)	$t \rightarrow h$	(P4)
(8)	h	(6)+(7), by modus ponens

Therefore, the argument is valid.