Week 8

Recap

<u>Sets and predicates</u>

Given a domain D:

- each predicate $P(\cdot)$ with a variable in D corresponds to a set: $A = \{x : P(x) \equiv T\}$.

Example: $D = \mathbb{Z}$, $P(x) \equiv$ "$x \bmod 2 = 0$". Then, $A = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$

i.e. A is the set of even numbers.

- each set $A \subseteq D$ corresponds to a predicate:

$P(x) \equiv x \in A$.

Example: $D = \mathbb{Z}$, $A = \{\cdots, -5, -2, 1, 4, 7, 10, \cdots\}$.

Then, $P(x) \equiv$ "$x \bmod 3 = 1$".

This "duality" gives rise to <u>notions in set theory</u>.

$$A \subseteq B \equiv \forall x, (x \in A) \to (x \in B)$$

$$\begin{aligned} A = B &\equiv \forall x, (x \in A) \leftrightarrow (x \in B) \\ &\equiv \forall x, ((x \in A) \to (x \in B)) \wedge ((x \in B) \to (x \in A)) \\ &\equiv (A \subseteq B) \wedge (B \subseteq A) \end{aligned}$$

$A \cup B = \{x : (x \in A) \vee (x \in B)\}$ union

$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$ intersection

$\overline{A} = \{x : \neg(x \in A)\}$ complement

$A - B = A \cap \overline{B} = \{x : (x \in A) \wedge \neg(x \in B)\}$ set difference

$(A \setminus B)$

$\emptyset \quad \forall x,\ x \in \emptyset \equiv F$ empty set

Further notions in set theory

Cardinality $|A|$ is the number of elements in A

Power set $S \in \mathbf{P}(A)$ iff $S \subseteq A$

Cartesian product $(a,b) \in \mathbf{A} \times \mathbf{B} \equiv (a \in A) \wedge (b \in B)$

e.g. $\mathbb{R} \times \mathbb{R} (= \mathbb{R}^2)$ is the two-dimensional plane

Cardinality formulas (related to combinatorics)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|P(A)| = 2^{|A|} = \sum_{j=0}^{|A|} \binom{|A|}{j}$$

$$|A \times B| = |A| \cdot |B|$$

Proving set identities

Method 1 : showing $A = B$ by first showing $A \subseteq B$ and then showing $B \subseteq A$ (typically used in theoretical math)

Step 1: proving $\forall x, (x \in A) \rightarrow (x \in B)$.

Step 2: proving $\forall x, (x \in B) \rightarrow (x \in A)$.

You will often need to perform case-by-case analysis.

Method 2 : applying set identity theorems

Completely analogous to proving logical equivalence using logical equivalence laws. In fact, every set identity law in the lecture except $A - B = A \cap \overline{B}$ (this one is by definition) is a direct translation of a logical equivalence law.

Method 3 : membership table (recommended for exams)

Completely analogous to proving logical equivalence using a truth table. The idea is to let $p \equiv (x \in A)$, $q \equiv (x \in B)$, $r \equiv (x \in C)$, ...

Q6: Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Proof of "$\Leftarrow$": assume that $A \subseteq B$. Let $S \in P(A)$ be arbitrary. Then, $S \subseteq A \subseteq B$ and thus $S \subseteq B$. This means that $S \in P(B)$. Therefore, $P(A) \subseteq P(B)$. We have proved that $A \subseteq B \Rightarrow P(A) \subseteq P(B)$.

Proof of "$\Rightarrow$": assume that $P(A) \subseteq P(B)$. Since $A \subseteq A$, we get $A \in P(A) \subseteq P(B)$ and thus $A \in P(B)$. This means that $A \subseteq B$. We have proved that $P(A) \subseteq P(B) \Rightarrow A \subseteq B$.

Therefore, we conclude that $P(A) \subseteq P(B) \iff A \subseteq B$.

Q9: Prove that for the sets A, B, C, D

$$\underbrace{(A \times B) \cup (C \times D)}_{LHS} \subseteq \underbrace{(A \cup C) \times (B \cup D)}_{RHS}.$$

Does the equality hold?

Let $(x, y) \in LHS$ be arbitrary. Then, either $(x, y) \in A \times B$ or $(x, y) \in C \times D$.

Case 1: $(x, y) \in A \times B$. In this case, $x \in A$ and $y \in B$. Subsequently, $x \in A \cup C$, $y \in B \cup D$, and thus $(x, y) \in (A \cup C) \times (B \cup D)$.

Case 2: $(x, y) \in C \times D$. In this case, $x \in C$ and $y \in D$. Subsequently, $x \in A \cup C$, $y \in B \cup D$, and thus $(x, y) \in (A \cup C) \times (B \cup D)$.

In both cases, $(x, y) \in RHS$. Therefore, $LHS \subseteq RHS$.

The equality does not hold, as demonstrated by the following counterexample.

Let $A = B = \{0\}$, $C = D = \{1\}$. Then,

$LHS = \{(0,0)\} \cup \{(1,1)\} = \{(0,0), (1,1)\}$,

$RHS = \{0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\}$.

$LHS \neq RHS$.

Q11: How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.

The number of subsets of $\{1, \cdots, n\}$ is $|P(\{1, \cdots, n\})| = 2^n$.

If n is odd: for every $S \subseteq \{1, \cdots, n\}$ with $|S|$ even, $\bar{S} = \{1, \cdots, n\} - S$ satisfies $\bar{S} \subseteq \{1, \cdots, n\}$ and $|\bar{S}| = n - |S|$, which means $|\bar{S}|$ is odd (since n is odd). This analysis shows that every subset of $\{1, \cdots, n\}$ with an even number of elements corresponds to exactly one subset of $\{1, \cdots, n\}$ with an odd number of elements, i.e. the number of subsets of $\{1, \cdots, n\}$ with an even number of elements and the number of subsets of $\{1, \cdots, n\}$ with an odd number of elements are the same, and they are both equal to $\frac{2^n}{2} = 2^{n-1}$.

Example: when $n = 3$,

even	odd
$\emptyset$	$\{1,2,3\}$
$\{1,2\}$	$\{3\}$
$\{1,3\}$	$\{2\}$
$\{2,3\}$	$\{1\}$

This also follows from the fact that $\forall\, 0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$.

$$\sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k} = \sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \underbrace{\binom{n}{n-k}}_{(n-k) \text{ is odd}} = \sum_{\substack{0 \le j \le n \\ j \text{ is odd}}} \binom{n}{j}.$$

If n is even: a set $S \subseteq \{1, \cdots, n\}$ with $|S|$ even either contains n, or does not contain n.

Case 1: $n \in S$. In this case, in order to build such an S, one first chooses $T \subseteq \{1, \cdots, n-1\}$ with $|T|$ odd, and then let $S = T \cup \{n\}$. Since $n-1$ is odd, there are 2^{n-2} ways to choose such a T, and thus there are 2^{n-2} ways to build an S satisfying $S \subseteq \{1, \cdots, n\}$, $|S|$ is even, and $n \in S$.

Case 2: $n \notin S$. In this case, in order to build such an S, one chooses $S \subseteq \{1, \cdots, n-1\}$ with $|S|$ even. Since $n-1$ is odd, there are 2^{n-2} ways to do so.

In total, there are $2^{n-2} + 2^{n-2} = 2^{n-1}$ such subsets.

Alternatively, since the number of subsets of $\{1, \cdots, n\}$ with k elements ($0 \le k \le n$) is $\binom{n}{k}$, our goal is to compute $\sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k}$.

We know that:

$$2^n = (1+1)^n = \sum_{0 \le k \le n} \binom{n}{k} = \left[\sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k}\right] + \left[\sum_{\substack{0 \le j \le n \\ j \text{ is odd}}} \binom{n}{j}\right],$$

$$0 = (1-1)^n = \sum_{0 \le k \le n} \binom{n}{k}(-1)^k = \left[\sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k}\right] - \left[\sum_{\substack{0 \le j \le n \\ j \text{ is odd}}} \binom{n}{j}\right].$$

Thus, $\sum_{\substack{0 \le k \le n \\ k \text{ is even}}} \binom{n}{k} = \frac{2^n}{2} = 2^{n-1}$.

Q12: Prove the following set equality:

$$\{7a + 9b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

It is clear that every element in $\{7a+9b \mid a, b \in \mathbb{Z}\}$ is an integer. Thus, LHS $\subseteq$ RHS. It remains to show that RHS $\subseteq$ LHS.

Let $n \in \mathbb{Z}$ be arbitrary. Since $7 \cdot (-5) + 9 \cdot 4 = -35 + 36 = 1$, $7 \cdot (-5n) + 9 \cdot (4n) = n$. Therefore, $n \in \{7a+9b \mid a, b \in \mathbb{Z}\}$.

We have proved that RHS $\subseteq$ LHS and we conclude that LHS = RHS.

Q14: For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \overline{A} \cup B.$$

using set identities.

$$
\begin{aligned}
LHS &= \overline{(A - B) - (B - C)} \\
&= \overline{(A \cap \overline{B}) \cap \overline{B - C}} && \text{(alternative representation of set difference)} \\
&= \overline{(A \cap \overline{B}) \cap \overline{B \cap \overline{C}}} && \text{(alternative representation of set difference)} \\
&= \overline{(A \cap \overline{B}) \cap (\overline{B} \cup C)} && \text{(De Morgan)} \\
&= \overline{(A \cap \overline{B} \cap \overline{B}) \cup (A \cap \overline{B} \cap C)} && \text{(distributivity)} \\
&= \overline{(A \cap \overline{B}) \cup ((A \cap \overline{B}) \cap C)} && \text{(idempotence)} \\
&= \overline{A \cap \overline{B}} && \text{(absorption)} \\
&= \overline{A} \cup B && \text{(De Morgan)} \\
&= RHS
\end{aligned}
$$

Additional exercise

Q3: How many positive integers less than 1000

1. are divisible by7?
2. are divisible by 7 but not by 11?
3. are divisible by both 7 and 11?
4. are divisible by either 7 or 11?
5. are divisible by exactly one of 7 and 11?
6. are divisible by neither 7 nor 11?

Let $U = \{k \in \mathbb{Z} : 1 \le k \le 999\}$. For every $m \in \mathbb{N}$, let

$S_m = \{k \in \mathbb{Z} : 1 \le k \le 999,\ k \text{ is divisible by } m\}$.

Observe that $|S_m| = \left\lfloor \frac{999}{m} \right\rfloor$ ($\lfloor \cdot \rfloor$ is the floor function)

1. $|S_7| = \left\lfloor \frac{999}{7} \right\rfloor = 142.$

2. Notice that $A - B = A - (A \cap B)$, and $A \cap B \subseteq A$. Therefore,

$|A - B| = |A - (A \cap B)| = |A| - |A \cap B|.$

$$|S_7 - S_{11}| = |S_7| - |S_7 \cap S_{11}| = |S_7| - |S_{77}| = 142 - \left\lfloor \frac{999}{77} \right\rfloor$$
$$= 142 - 12 = 130.$$

3. $|S_7 \cap S_{11}| = |S_{77}| = 12.$

4. $$|S_7 \cup S_{11}| = |S_7| + |S_{11}| - |S_7 \cap S_{11}| = 142 + \left\lfloor \frac{999}{11} \right\rfloor - 12$$
$$= 142 + 90 - 12 = 220.$$

5. $|S_{11} - S_7| = |S_{11}| - |S_{11} \cap S_7| = |S_{11}| - |S_{77}| = 90 - 12 = 78.$

$$|(S_7 - S_{11}) \cup (S_{11} - S_7)| = |S_7 - S_{11}| + |S_{11} - S_7| = 130 + 78 = 208.$$

Since $(S_7 - S_{11}) \cap (S_{11} - S_7) = \emptyset$.

6. $|U - (S_7 \cup S_{11})| = |U| - |S_7 \cup S_{11}| = 999 - 220 = 779.$

Additional Challenges

QUESTION 3. **(15 marks)**

How many numbers in the range [1812, 2020] (inclusive of both 1812 and 2020) are integer multiples of **exactly** one of the two factors 4 and 5? Justify your answer.

Let $S_m = \{k \in \mathbb{Z} : 1812 \le k \le 2020,\ k \text{ is divisible by } m\}$ for $m \in \mathbb{N}$.

Our goal is to compute:

$$|(S_4 - S_5) \cup (S_5 - S_4)|$$

$$= |S_4 - S_5| + |S_5 - S_4| \quad (\text{Since } (S_4 - S_5) \cap (S_5 - S_4) = \emptyset)$$

$$= |S_4 - (S_4 \cap S_5)| + |S_5 - (S_4 \cap S_5)|$$

$$= |S_4| - |S_{20}| + |S_5| - |S_{20}|$$

$$= |S_4| + |S_5| - 2|S_{20}|.$$

$S_m = T_m - R_m$ where

$T_m = \{k \in \mathbb{Z} : 1 \le k \le 2020,\ k \text{ is divisible by } m\}$,

$R_m = \{k \in \mathbb{Z} : 1 \le k \le 1811,\ k \text{ is divisible by } m\}$

Warning: $|S_m| \neq \left\lfloor \frac{2020 - 1811}{m} \right\rfloor$

e.g. $\left\lfloor \frac{2020-1811}{4} \right\rfloor = 52 \neq |S_4|$

$$|S_4| = \left\lfloor \frac{2020}{4} \right\rfloor - \left\lfloor \frac{1811}{4} \right\rfloor = 53,$$

$$|S_5| = \left\lfloor \frac{2020}{5} \right\rfloor - \left\lfloor \frac{1811}{5} \right\rfloor = 42,$$

$$|S_{20}| = \left\lfloor \frac{2020}{20} \right\rfloor - \left\lfloor \frac{1811}{20} \right\rfloor = 11.$$

Answer: $53 + 42 - 2 \cdot 11 = 73$.

$A \triangle C = (A - C) \cup (C - A)$, $B \triangle C = (B - C) \cup (C - B)$

Prove, for any sets A, B, and C, if $A \triangle C = B \triangle C$ then $A = B$.

Approach 1

Membership table & truth table

A	B	C	$A \triangle C$	$B \triangle C$	$(A \triangle C = B \triangle C)$	$(A = B)$
0	0	0	0	0	T	T
0	0	1	1	1	T	T
0	1	0	0	1	F	F
0	1	1	1	0	F	F
1	0	0	1	0	F	F
1	0	1	0	1	F	F
1	1	0	1	1	T	T
1	1	1	0	0	T	T

This table shows that $A \triangle C = B \triangle C$ iff $A = B$.

Approach 2

Proof by contrapositive. Suppose $A \neq B$. Then, there exists x such that either $x \in A$ and $x \notin B$, or $x \in B$ and $x \notin A$. Since the roles of A and B are symmetric, we assume without loss of generality that $x \in A$ and $x \notin B$.

Case 1: $x \in C$. Then, $x \in A \cap C$ and $x \notin A \triangle C$

But $x \in C - B \subseteq B \triangle C$. Thus, we have found $x \in B \triangle C$ and $x \notin A \triangle C$.

Case 2: $x \notin C$. Then, $x \in A - C \subseteq A \triangle C$. Since $x \notin B$, $x \notin C$, we get $x \notin B \triangle C$. Thus, we have found $x \in A \triangle C$ and $x \notin B \triangle C$.

In both cases, we can conclude that $A \triangle C \neq B \triangle C$.

Thus, by proof by contrapositive, if $A \triangle C = B \triangle C$, then $A = B$.

Approach 3

"$\triangle$" can be interpreted as "exclusive OR" (XOR), i.e.

$S \triangle T = \{x : (x \in S) \text{ XOR } (x \in T)\}$. Some consequences:

(1) $S \triangle T = \phi \iff S = T$;

(2) $S \triangle \phi = S$, $S \triangle S = \phi$;

(3) $\triangle$ is associative, i.e. $(R \triangle S) \triangle T = R \triangle (S \triangle T)$.

Solution: $A \triangle C = B \triangle C \iff (A \triangle C) \triangle (B \triangle C) = \phi$

$\iff A \triangle C \triangle C \triangle B = \phi$

$\iff A \triangle (C \triangle C) \triangle B = \phi$

$\iff (A \triangle \phi) \triangle B = \phi$

$\iff A \triangle B = \phi \iff A = B$.